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Orientable quadrilateral embeddings of products of graphs

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Abstract

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It is proved that for each connected regular graph G there exists an integer $m>0$ such that for each integer $n \geqslant m$ the Cartesian product $G \times Q_n$ has an orientable quadrilateral embedding, thereby extending the result from bipartite to regular nonbipartite graphs.

1. Introduction

Gerhard Ringel [6] has proved that an arbitrary cube Q_n admits a quadrilateral embedding. Later this idea has been generalized by White [7, 9] who proved, among other things, that the Cartesian product of even cycles $C_{2n_1} \times C_{2n_2} \times \cdots \times C_{2n_r}$ admits a quadrilateral embedding. These ideas have been extended in a series of papers [3, 4]. In particular, it has been shown that the Cartesian product of two connected regular bipartite graphs of the same valence admits an orientable quadrilateral embedding. It has been shown that the Cartesian product of an arbitrary connected bipartite graph with a large cube Q_n admits an orientable quadrilateral embedding. A similar statement can be made for general, nonbipartite graphs. There, however, the construction yields a non-orientable embedding if and only if the graph contains an odd cycle. In this paper we use a different construction to determine an orientable quadrilateral embedding of the Cartesian product of an arbitrary regular graph with a large cube. All graphs in this paper are simplicial. The reader is referred to the books by Ringel, White, Gross and Tucker [5, 8, 1] for motivation and background in topological graph theory.

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2. Results and proofs

Lemma 2.1. *Let G be 1-factorable graph of valance r. Then $G \times Q_{2r}$ has an orientable quadrilateral embedding.*

Proof. Since Q_{2r} is bipartite each vertex $w = (v, u) \in V(G \times Q_{2r})$ can be called either *white* or *black* according to whether u is white or black, respectively. However this vertex coloring of $G \times Q_{2r}$ is proper only if G is bipartite.

Now color the edges of G with colors $1, 2, \ldots, r$. As G is 1-factorable this can always be achieved. This edge-coloring induces an orientable embedding of G determined by a cyclic permutation of colors $\pi = (1, 2, \ldots, r)$. The rotation ρ_v at each vertex $v \in V(G)$ is simply determined by π. In order to simplify notation we will write just $\rho_v = \pi$. On the union of boundaries of all faces of the embedding of G each edge appears exactly twice. Hence each 1-factor appears exactly twice. We mark arbitrarily one appearance of each edge by $+1$ and the other by -1. For each color i, $i = 1, 2, \ldots, r$ we denote by i^{+1} and i^{-1} the two appearances on the boundaries. Q_{2r} has $2r$ 1-factors that we get in an obvious way if we consider Q_{2r} as

$$\underbrace{K_2 \times K_2 \times \cdots \times K_2}_{2r \text{ times}}.$$

We denote the $2r$ 1-factors as follows: $\underline{1}^{+1}, \underline{2}^{+1}, \ldots, \underline{r}^{+1}, \underline{1}^{-1}, \underline{2}^{-1}, \ldots, \underline{r}^{-1}$. This gives us a one-to-one map from the $2r$ 1-factors of Q_{2r} to the $2r$ *appearances* of 1-factors in the embedding of G. It also gives us an edge coloring of $G \times Q_{2r}$ with the $3r$ colors $\underline{1}^{+1}, 1, \underline{1}^{-1}, \underline{2}^{+1}, 2, \underline{2}^{-1}, \ldots, \underline{r}^{+1}, r, \underline{r}^{-1}$.

Let v be an arbitrary vertex of G. We know that the rotation about v is determined by the cyclic permutation of colors π. If we consider appearances we get

$$(\underline{1}^{\varepsilon_1}, \underline{1}^{-\varepsilon_1}, \underline{2}^{-\varepsilon_2}, \underline{2}^{-\varepsilon_2}, \ldots, \underline{r}^{\varepsilon_r}, \underline{r}^{-\varepsilon_r}),$$

where $\varepsilon_i \in \{-1, +1\}$, $i = 1, 2, \ldots, r$. Clearly the exponents ε_i depend on v. Let $w = (v, u)$ be a vertex of $G \times Q_{2r}$ with first component v. Now consider the following rotation $\sigma = (\underline{1}^{\varepsilon_1}, 1, \underline{1}^{-\varepsilon_1}, \underline{2}^{\varepsilon_2}, 2, \underline{2}^{-\varepsilon_2}, \ldots, \underline{r}^{\varepsilon_r}, r, \underline{r}^{-\varepsilon_r})$. Finally we are in a position to define an embedding for $G \times Q_{2r}$. Each black vertex w gets rotation $\rho_w = \sigma$, each white one gets rotation $\rho_w = \sigma^{-1}$. It is easy to verify that this embedding is indeed quadrilateral. Moreover, it has some disjoint quadrilateral *patchworks*. For instance, for each i, the edges colored with i, i^+ form a quadrilateral patchwork. For a definition of a patchwork, see [1]; see also [3, 4]. □

Now we are in position to state and prove our main result.

Theorem 2.2. *Let G be a regular graph of valance r and girth at least four. Then for all numbers n such that $n > 2r + 2$, the graph $G \times Q_n$ has an orientable quadrilateral, genus embedding.*

Proof. According to Kotzig, [2] the graph $H = G \times K_2$ is $(r+1)$-valent and 1-factorable. By our lemma, $G \times K_2 \times Q_{2r+2}$ has a quadrilateral, orientable embedding. Using the proof of our lemma we see that it also has a patchwork. Therefore we may multiply it repeatedly by K_2 and obtain each time an orientable quadrilateral embedding. This embedding is clearly a genus embedding since the girth of the resulting graph is 4. This means the theorem holds for any $n > 2r + 2$. □

There are three problems that we are interested in.

- If G and H are arbitrary r-valent bipartite, connected graphs, we know how to calculate the genus of their Cartesian product. It would be nice if we could find a quadrilateral, orientable embedding of the $G \times H$ for arbitrary r-valent, 1-factorable graphs, (or find a counterexample).
- Is it possible to extend our theorem so that we take the Cartesian product of even cycles $C_{2n_1} \times C_{2n_2} \times \cdots \times C_{2n_r}$ instead of large cubes Q_n?
- Now we know that for large n the graph $G \times Q_n$ has an orientable quadrilateral embedding for connected G that is regular or bipartite. Is this statement true even for nonregular nonbipartite graphs G?

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